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Accurate Efficient Evaluation of Bessel Transform; Programs and Error Analysis

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Preface

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The method of Filon numerical integration for Fourier transforms is extended to Bessel transforms of ~~the~~ a certain form

$$G(\omega) = \int_0^\infty dx \; J_0(\omega x) \; g(x) \; ,$$

for general g(x). Specifically, for the two cases where g(x) is approximated by

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18. SUBJECT TERMS (Cont'd.)

Panel Width
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19. ABSTRACT (Cont'd.)

~~(a)~~ straight lines, or

~~(b)~~ parabolas,

over abutting panels, the corresponding integrals in the Bessel transform $G(\phi)$ are evaluated exactly (within computer round-off error). Although these integrals cannot be expressed in closed form (as for Filon's case), a recursive procedure and an asymptotic expansion yield rapid accurate evaluation of the required quantities.

Programs are furnished for both cases ~~(a)~~ and (b) in BASIC. Furthermore, two versions of each are furnished: a faster one requiring considerable storage, and a slower one requiring very little storage. The presence and location of aliasing is predicted and its magnitude is investigated numerically. The error dependence on the panel width used in both cases (a) and (b) is established by means of numerical examples, one with a very fast decay with ω, the other with a very slow decay with ϕ. Comparisons with standard Trapezoidal and Simpson's rules reveal that the new procedures are error <u>maintenance</u> procedures, tending to keep the absolute error for larger ω comparable to that near $\omega = 0$, whereas the standard rules are subject to aliasing errors that become very significant for larger ω.

Extensions to more general Bessel transforms are possible and procedures for obtaining them are outlined.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

x	variable of integration, (2)
$g(x)$	function to be transformed, (2)
$J_0(u)$	zero-th order Bessel function, (2)
ω	transform variable, (2)
$G(\omega)$	Bessel transform of $g(x)$, (2)
x_ℓ	lower limit of integral, (11)
x_r	upper limit of integral, (11)
h	panel width, sampling increment, (12)
x_n	general sample point in x, figure 1
g_n	sample value $g(x_n)$, figure 1
$A(u)$	indefinite integral of J_0, (13)
$J_1(u)$	first order Bessel function, (14)
$B_0(u)$	auxiliary function, (16)
$B_1(u)$	auxiliary function, (17)
I_n	integral contribution, (18)
y	normalized variable, (19)
ℓ, r	left and right integer limits, (20)
θ	product ωh, (22)
Δ	sampling increment in ω, (25)
k	integer location of sample in ω, (25)
K_1, K_2	limits on k, (25)
S_ℓ, S_r	auxiliary quantities, (31)
Q_ℓ, Q_r	auxiliary quantities, (31)
D_n, F_n, R_n	auxiliary sequences, (32)

LIST OF SYMBOLS (Cont'd)

a(b)c	sequence a, a+b, a+2b, ..., c-b, c, (33)
$\Sigma\!\!\!\!\!+$	Summation over every other term, (30)
$I_0(u)$	zero-th order modified Bessel function, (40)

ACCURATE EFFICIENT EVALUATION OF BESSEL TRANSFORM; PROGRAMS AND ERROR ANALYSIS

INTRODUCTION

The method of Filon integration for Fourier transforms [1], [2; pages 408-409], [3; pages 67-75], [4; page 400], [5; page 890], [6; pages 62-66], [7]

$$\int_{-\infty}^{+\infty} dx \exp(i\omega x)\, g(x) \tag{1}$$

is well established and very useful for accurate numerical work. Instead of the standard Simpson's rule, which would approximate the complete integrand $\exp(i\omega x)\, g(x)$ by parabolas over abutting pairs of panels, Filon's method approximates only the function $g(x)$ by parabolas, and carries out the corresponding integrals in (1) analytically. These closed form integrals are then evaluated with computer aid. Since the exponential in (1) is being handled exactly for all ω, the hope is that the error of approximating (1) by means of Filon's method will be substantially the same for larger ω as for small ω (where all the error arises from approximating $g(x)$). That is, Filon's method is expected to be an error maintenance procedure, whereby the absolute error does not increase significantly with ω. Certainly that is not the case for the Trapezoidal and Simpson rules, where significant aliasing severely limits the accuracy of the results for larger ω.

An alternative simpler procedure to Filon's method for Fourier transforms is to approximate g(x) by straight lines over abutting panels, and again to evaluate the resultant integrals in (1) analytically in closed form. This (less-accurate) procedure is documented in [8; pages 418-419], for example.

Here, we will extend these two procedures to a Bessel transform of the form

$$G(\omega) = \int_0^\infty dx\ J_0(\omega x)\ g(x)\ , \tag{2}$$

where g(x) is an arbitary given function, and J_0 is the zeroth-order Bessel function. One of the major differences we encounter, relative to Filon's method, is that the resultant integrals cannot all be evaluated in closed form. In order to circumvent this problem, we use a combination of a downward recursion and an asymptotic expansion, which are limited in accuracy only by the inherent round-off error of the computer utilized, thereby obtaining an efficient useful procedure for numerical evaluation of the pertinent integrals and functions.

To give a physical application where the Bessel transform arises, consider that we are interested in two-dimensional Fourier transform

$$\iint_{-\infty}^{+\infty} dx\ dy\ \exp(iux + ivy)\ f_2(x,y)\ , \tag{3}$$

where function f_2 has isotropic behavior. That is, suppose the dependence of f_2 is solely on the distance from the origin of coordinates:

$$f_2(x,y) = f_1(\sqrt{x^2 + y^2})\ . \tag{4}$$

Then (3) becomes

$$\iint_{-\infty}^{+\infty} dx\, dy\, \exp(iux + ivy)\, f_1(\sqrt{x^2 + y^2}) =$$

$$= 2\pi \int_0^{\infty} dr\, J_0(\omega r)\, r\, f_1(r) \;, \tag{5}$$

where we changed to cylindrical coordinates and have defined

$$\omega = (u^2 + v^2)^{1/2} \;. \tag{6}$$

Thus, (5) is of the form of (2), upon identification of g(x) as x $f_1(x)$.

Suppose in (3) that the f_2 dependence on x,y is more general than (4), namely of the form

$$f_2(x,y) = f_1\left(\left[\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2 - 2\rho\left(\frac{x - x_0}{a}\right)\left(\frac{y - y_0}{b}\right)\right]^{1/2}\right) \;, \tag{7}$$

which allows for a general center point of symmetry x_0, y_0, as well as a tilted elliptical shape. Then substitution in (3) yields, after a cylindrical coordinate change, the result

$$\frac{2\pi ab}{(1 - \rho^2)^{1/2}} \exp(iux_0 + ivy_0) \int_0^{\infty} dr\, J_0(\omega r)\, r\, f_1(r) \;, \tag{8}$$

where now

$$\omega = \left[\frac{a^2u^2 + b^2v^2 + 2\rho abuv}{1 - \rho^2}\right]^{1/2} \;. \tag{9}$$

Again, the fundamental Bessel transform of the form of (2) results, where $g(x)$ is $x\ f_1(x)$.

On the other hand, if $G(\omega)$ is specified in (2) for $\omega > 0$, the corresponding solution to this integral equation is

$$g(x) = x \int_0^\infty d\omega\ J_0(x\omega)\ \omega\ G(\omega)\ , \qquad (10)$$

which is again a Bessel transform of the form of (2).

Thus, we have presented several instances where the transform given by (2) is of interest and must be accomplished accurately for large as well as small arguments of the transform variable ω.

LINEAR APPROXIMATION

The integral of interest here is

$$G(\omega) = \int_{x_\ell}^{x_r} dx\; J_0(\omega x)\; g(x) \;, \tag{11}$$

where left-end point x_ℓ could be zero, and right-end point x_r could be taken so large that g(x) is essentially zero for $x > x_r$. (If x_ℓ is negative, the values of g could be folded over to the positive x-axis, using $g(x) + g(-x)$ as the new integrand, since $J_0(\omega x)$ is even in x.) We break interval x_ℓ, x_r into a number of abutting panels, each of the same width h, and fit g(x) by straight lines over each of those panels. The fits for the left-end point and an abutting (internal) point x_n are depicted in figure 1, where it is temporarily presumed that the adjacent sample values of

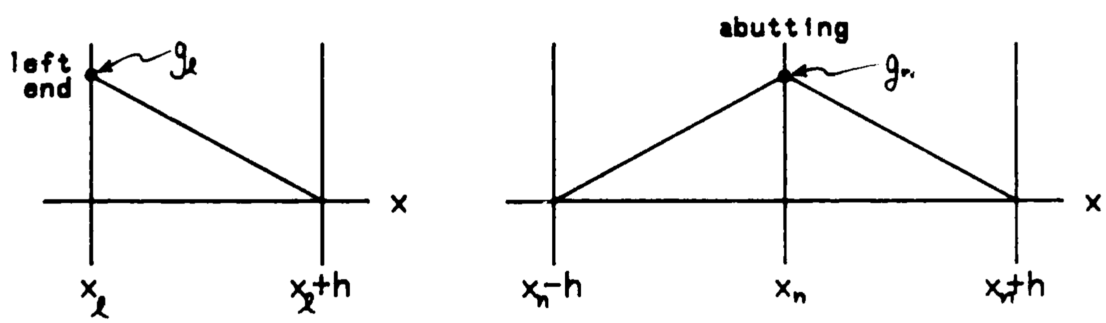

Figure 1. Linear Approximations to g(x)

function g(x) are zero; this allows us to isolate the contribution of each sample of g(x) to the total desired in (11). The straight lines pass through the function value $g_n = g(x_n)$ at sample value x_n, and are zero at the adjacent sample points. h is the sampling increment in x applied to g(x). The situation at the right end is the mirror image of that at the left end, depicted in figure 1.

If ω is zero in (11), the approximation afforded to the integral by means of figure 1 is obviously

$$G(0) \cong h\left[\frac{1}{2}\, g_{\ell} + g_{\ell+1} + \ldots + g_{r-1} + \frac{1}{2}\, g_r\right] =$$

$$= h\left[\frac{1}{2}\, g(x_{\ell}) + g(x_{\ell+1}) + \ldots + g(x_{r-1}) + \frac{1}{2}\, g(x_r)\right] \quad \text{for } \omega = 0 \,, \tag{12}$$

which is just the Trapezoidal rule. For $\omega > 0$, considerably more effort is required; there is no need to consider $\omega < 0$, since $J_0(\omega x)$ is even in ω. Before we get into that derivation, we must introduce some auxiliary functions.

SPECIAL FUNCTION DEFINITIONS

Define the integral

$$A(u) = \int_0^u dt\; J_0(t) \; . \tag{13}$$

This function cannot be evaluated in closed form; a table of A(u) is available in [5; pages 492-493]. On the other hand, the integral

$$\int_0^u dt\ t\ J_0(t) = u\ J_1(u) \tag{14}$$

is immediately available by use of [5; 9.1.30]. And two integrations by parts, coupled with (13), yields the result

$$\int_0^u dt\ t^2\ J_0(t) = u^2\ J_1(u) + u\ J_0(u) - A(u)\ . \tag{15}$$

We will also find use for the auxiliary functions

$$B_0(u) \equiv A(u) - u\ J_0(u) = \int_0^u dt\ t\ (u - t)\ J_0(t)\ , \tag{16}$$

and

$$B_1(u) \equiv A(u) - J_1(u) = \int_0^u dt\ \left(1 - \frac{t}{u}\right) J_0(t)\ . \tag{17}$$

All of these functions, A, B_0, B_1, are zero at the origin and are odd. Numerical evaluation of these functions is considered in appendix A.

ABUTTING POINT

For an abutting (internal) point x_n in the interval (x_ℓ, x_r), as depicted on the right-hand side of figure 1, the contribution to integral (11), due to this single sample point $g_n = g(x_n)$, is

$$I_n = \int_{x_n-h}^{x_n} dx\ J_o(\omega x)\ g_n\ (1 + y) + \int_{x_n}^{x_n+h} dx\ J_o(\omega x)\ g_n\ (1 - y)\ , \qquad (18)$$

where we have defined

$$y = \frac{x - x_n}{h}\ . \qquad (19)$$

We now assume that the n-th sample point x_n is taken such that

$$x_n = n\ h \quad \text{for } \ell \le n \le r\ . \qquad (20)$$

This makes

$$x_\ell = \ell h, \quad x_r = rh, \quad \text{i.e. } \frac{x_r}{x_\ell} = \frac{r}{\ell} = \text{rational}\ . \qquad (21)$$

This constitutes a restriction on ratio x_r/x_ℓ in (11); it has been adopted here in order to minimize the number of calculations of the Bessel function J_o later, when we consider the multiple values of ω desired for (11). (The procedure presented here can be extended to the general case where x_ℓ is arbitrary and $x_n = x_\ell + nh$, if desired.) If x_ℓ is zero, then the choice in (20) is no restriction at all.

APPROXIMATION TO INTEGRAL

An important parameter in this numerical integration procedure is the quantity

$$\Theta = \omega h \tag{22}$$

which is the product of "radian frequency" ω and the sampling increment h. As we shall see, values of Θ near π and 2π will constitute points of considerable aliasing; see [4; page 400] for a discussion of the Fourier transform case.

When the procedure in (18)-(19) is extended to include the left-end and right-end points of integral (11), and the various integrals evaluated with the help of (13)-(17), the total approximation is given by appendix B in several alternative forms, one of which is (B-7):

$$\begin{aligned} \omega\, G(\omega) \cong & \left[\ell g_{\ell+1} - (\ell + 1) g_\ell\right] B_1(\ell\Theta) - g_\ell\, J_1(\ell\Theta) + \\ & + \left[r\, g_{r-1} - (r - 1) g_r\right] B_1(r\Theta) + g_r\, J_1(r\Theta) + \\ & + \sum_{n=\ell+1}^{r-1} n\left[g_{n+1} - 2g_n + g_{n-1}\right] B_1(n\Theta) \end{aligned} \tag{23}$$

where

$$g_n = g(x_n) = g(nh) \ . \tag{24}$$

Reasons for this grouping of terms, including speed of execution and storage requirements, are discussed below.

SAMPLING INCREMENT FOR ω

When output variable ω in integral (11) is restricted to multiples of a sampling increment Δ, according to

$$\omega = k\Delta \quad \text{for } 1 \le K_1 \le k \le K_2 \ , \tag{25}$$

then $n\Theta = nk\Delta h$, meaning that the arguments of the $B_1(u)$ function in (23) are limited to integer multiples of $h\Delta$, the product of the sampling increment in input variable x and the sampling increment in output (transform) variable ω. The explicit relationship for $G(\omega) = G(k\Delta)$ is given by specializing (23) to the values (24), thereby obtaining

$$\begin{aligned} k\Delta\, G(k\Delta) \cong & \left[\ell\, g_{\ell+1} - (\ell + 1) g_\ell\right] B_1(\ell k\Delta h) - g_\ell\, J_1(\ell k\Delta h) + \\ & + \left[r\, g_{r-1} - (r - 1) g_r\right] B_1(rk\Delta h) + g_r\, J_1(rk\Delta h) + \\ & + \sum_{n=\ell+1}^{r-1} n\left[g_{n+1} - 2g_n + g_{n-1}\right] B_1(nk\Delta h) \quad \text{for } k \ge 1 \ . \end{aligned} \tag{26}$$

COMPUTATION TIME CONSIDERATIONS

Thus, we need evaluate $B_1(u)$ only at $u = m\Delta h$, where m is an integer. Furthermore, not all values of integer m will be encountered as n and k sweep out their respective values given by (20) and (25). And since $B_1(u)$, defined in (17) and (13), is the most time-consuming aspect of the computation of (26), it behooves us <u>not</u> to compute $B_1(m\Delta h)$ at values of m that will not be encountered, and <u>not</u> to recompute $B_1(m\Delta h)$ at values of m

that are encountered more than once. This latter situation arises when m is highly composite; for example, m = 12 = 4*3 = 6*2 = 12*1 could be encountered several times as n and k vary in (26).

In order to incorporate this time-saving feature into the Bessel integral evaluations required by (26), the values of $B_1(nk\Delta h)$ are computed only once and stored in a one-dimensional array at linear location m = nk. Unfortunately, this speed-up feature is achieved at the expense of considerable storage, for if n and k range up to N and K, respectively, the one-dimensional storage array must have NK cells, of which most are empty when N and K are large.

When N and K are so large that storage is not feasible, such as when x_r in (11) is large, and large ω is desired in (25), then the alternative procedure of direct brute-force evaluation of (26) for $B_1(nk\Delta h)$, repeated as often as necessary, but without storage, is employed. Recomputation of $B_1(m\Delta h)$ for some m values occurs, but evaluation at unused m values never does.

Thus we have two alternatives and two corresponding programs for (26): one faster routine which may require considerable storage, and a slower procedure utilizing very little storage. The former is recommended when feasible, while the latter furnishes a back-up position. Programs for both procedures are listed in appendix B.

BEHAVIOR FOR SMALL Θ

When Θ is small, differences of functions w th similar values are required in (23), as may be observed by the linear ω dependence on the left-side. The appropriate series development for this linear approximation approach to (11) is given in (B-11)-(B-12), through order Θ^2. Additional terms to order Θ^4, Θ^6 can be derived by extending the approach given there; however, an easier technique will be developed in the next section.

PARABOLIC APPROXIMATION

The integral of interest is again

$$G(\omega) = \int_{x_\ell}^{x_r} dx \; J_0(\omega x) \; g(x) \quad . \tag{27}$$

However, now we approximate g(x) by parabolas over abutting pairs of panels, each of width h. The fits for a mid-point, an abutting point, the left-end point, and the right-end point are illustrated in figure 2. Again, the

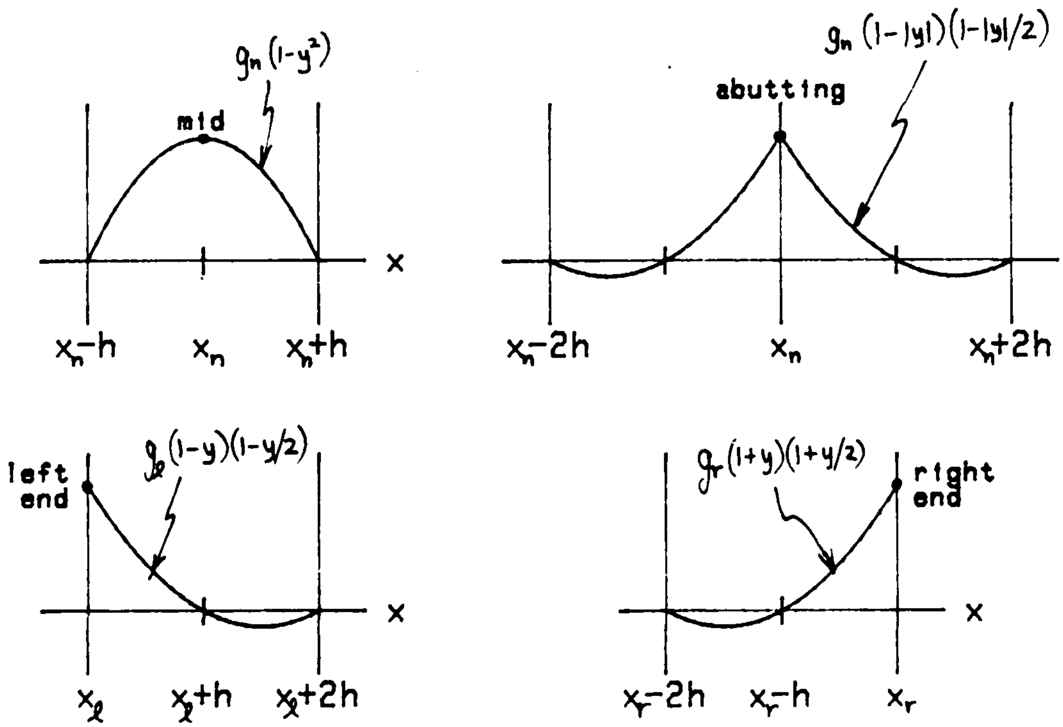

Figure 2. Parabolic Approximations to g(x)

contribution of each sample value $g_n = g(x_n)$ is isolated, by temporarily presuming that the neighboring sample values of g(x) are zero. The variable y in figure 2 is again the normalized quantity

$$y = \frac{x - x_n}{h} , \tag{28}$$

where h is the sampling increment in x applied to g(x).

If ω is zero in (27), the approximation afforded to the integral by means of figure 2 is

$$G(0) \cong \frac{h}{3}\left[g_\ell + 4g_{\ell+1} + 2g_{\ell+2} + \ldots + 2g_{r-2} + 4g_{r-1} + g_r\right] =$$
$$= \frac{h}{3}\left[g(x_\ell) + 4g(x_{\ell+1}) + 2g(x_{\ell+2}) + \ldots + 2g(x_{r-2}) + 4g(x_{r-1}) + g(x_r)\right] , \tag{29}$$

which is Simpson's rule.

APPROXIMATION TO INTEGRAL

Since J_o in (27) is even in ω, we only need to consider $\omega > 0$ in the following. The derivation of the approximation to integral (27), by means of the parabolic fits in figure 2, is carried out in appendix C, culminating in (C-10)-(C-12):

$$2\omega\ G(\omega) \cong \frac{1}{\theta^2} S_\ell\ B_o(\ell\theta) - Q_\ell\ B_1(\ell\theta) - 2g_\ell\ J_1(\ell\theta) -$$

$$- \frac{1}{\theta^2} S_r\ B_o(r\theta) + Q_r\ B_1(r\theta) + 2g_r\ J_1(r\theta) +$$

$$+ \frac{1}{\theta^2} \sum_{n=\ell+2}^{r-2} D_n\ B_o\ (n\theta) - \sum_{n=\ell+2}^{r-2} R_n\ B_1\ (n\theta)\ . \tag{30}$$

The auxiliary sequences utilized in (30) are defined below:

$$S_\ell = g_{\ell+2} - 2g_{\ell+1} + g_\ell$$

$$S_r = g_r - 2g_{r-1} + g_{r-2}$$

$$Q_\ell = \ell(\ell + 1)g_{\ell+2} - 2\ell(\ell + 2)g_{\ell+1} + (\ell + 2)(\ell + 1)g_\ell$$

$$Q_r = (r - 2)(r - 1)g_r - 2r(r - 2)g_{r-1} + r(r - 1)g_{r-2} \tag{31}$$

and

$$\left.\begin{aligned} D_n &= g_{n+2} - 2g_{n+1} + 2g_{n-1} - g_{n-2} \\ F_n &= g_{n+2} - 4g_{n+1} + 6g_n - 4g_{n-1} + g_{n-2} \\ R_n &= n^2 D_n + nF_n \end{aligned}\right\} \begin{aligned} &\text{for } n = \\ &(\ell + 2)(2)(r - 2)\ . \end{aligned} \tag{32}$$

The functions $B_o(u)$ and $B_1(u)$ are those defined in (13)-(17), and the slash on the summation symbol in (30) denotes skipping every other term, after starting at $n = \ell + 2$. A shorthand notation that will be used here is

$$n = \ell + 2,\ \ell + 4\ , \ldots ,\ r - 4,\ r - 2 = (\ell + 2)(2)(r - 2)\ . \tag{33}$$

Several important observations should be made about the result in (30)-(32). The four quantities in (31) are evaluated only once at the end points $n = \ell$ and r. The sequences in (32) must be evaluated at all the points listed in (33), that is, at every <u>other</u> interior point. All of these computations should be done once and stored, when given the function g(x), the limits x_ℓ, x_r, and sampling increment h, prior to ever considering which ω values will be of interest in (30). Input function g(x) must be evaluated at all $x = x_n = nh$ for $n = \ell(1)r$.

The time-consuming calculations of $B_0(u)$ and $B_1(u)$ in (30) are only necessary at the values $u = n\Theta$ for $n = \ell(2)r$, and need not be evaluated at any of the in-between points $n = (\ell + 1)(2)(r - 1)$. The Bessel function $J_1(u)$ need only be evaluated at end points $u = \ell\Theta$ and $r\Theta$; however, this quantity shows up as a free by-product of evaluating $B_0(u)$ and $B_1(u)$, by the method indicated in appendix A.

SAMPLING INCREMENT FOR ω

When output variable ω in desired integral (27) is restricted to multiples of a sampling increment Δ, according to

$$\omega = k\Delta \quad \text{for} \quad 1 \leq K_1 \leq k \leq K_2 , \tag{34}$$

then

$$\Theta = \omega h = k\Delta h , \tag{35}$$

and (30) takes on the form

$$2k\Delta\ G(k\Delta) \cong$$

$$= \frac{1}{(k\Delta h)^2} S_\ell\ B_o(\ell k\Delta h) - Q_\ell\ B_1(\ell k\Delta h) - 2g_\ell\ J_1(\ell k\Delta h) -$$

$$- \frac{1}{(k\Delta h)^2} S_r\ B_o(rk\Delta h) + Q_r\ B_1(rk\Delta h) + 2g_r\ J_1(rk\Delta h) +$$

$$+ \frac{1}{(k\Delta h)^2} \sum_{n=\ell+2}^{r-2} D_n\ B_o(nk\Delta h) - \sum_{n=\ell+2}^{r-2} R_n\ B_1(nk\Delta h)\ . \qquad (36)$$

At this point, the discussion in the sequel to (26) is directly relevant and should be reviewed. The only change in the presentation is to replace $B_1(u)$, there, by both $B_o(u)$ and $B_1(u)$ here. We again end up with two alternatives and two corresponding programs for evaluation of (36): one faster routine which may require considerable storage, and a slower procedure utilizing very little storage. Programs for both procedures are listed in appendix C.

BEHAVIOR FOR SMALL Θ

When Θ is small, differences of functions with similar values are required in (30), as may be observed by the linear ω dependence on the left side and the $1/\Theta^2$ dependence on the right side. This behavior is also typical for Filon's method, and indicates the need for a series expansion in powers of Θ for the right-hand side of (30) when Θ is small; see [5; (25.4.53)], for example. The appropriate series development for this parabolic approximation approach to (27) is given in (C-15)-(C-17), through order Θ^2. Additional terms to order Θ^4, Θ^6 can be derived by an obvious extension of the approach given there.

EXAMPLES

Two examples will be considered in this section; the first is a Rayleigh function,

$$g(x) = x \exp(-x^2/2) \quad \text{for } x > 0 \ , \tag{37}$$

for which Bessel transform (11) is [9; 6.631 4]

$$G(\omega) = \exp(-\omega^2/2) \ . \tag{38}$$

The second is a Gaussian function,

$$g(x) = \exp(-x^2) \quad \text{for } x > 0 \ , \tag{39}$$

leading to [9; 6.618 1]

$$G(\omega) = 1/2\sqrt{\pi} \exp(-\omega^2/8) \ I_0(\omega^2/8) \ . \tag{40}$$

These two examples are very different, in that transform (38) decays very quickly for large ω, whereas (40) decays very slowly for large ω. In fact, for the latter case [5; 9.7.1],

$$G(\omega) \sim 1/\omega \quad \text{as } \omega \to +\infty \ . \tag{41}$$

This difference will enable us to investigate both absolute and relative errors of the approximate numerical integration procedures developed earlier, over a wide range of values of ω.

ALIASING

The Bessel function J_o is rather similar to a sinusoid; in fact, for large z [9; 9.2.1],

$$J_o(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi}{4}\right) \text{ as } z \to +\infty. \tag{42}$$

Then when argument x in transform (11) is sampled at increment h, we encounter the behavior

$$J_o(\omega x_n) = J_o(\omega h n) = J_o(\Theta n) \sim \left(\frac{2}{\pi \Theta n}\right)^{1/2} \cos\left(\Theta n - \frac{\pi}{4}\right) \tag{43}$$

for large Θn. Now when $\Theta = 2\pi$, the cosine yields the same values as for $\Theta = 0$; this leads us to expect larger errors for the numerical integration procedure when Θ is near 2π.

For a Fourier transform, this aliasing effect was studied quantitatively in [10; appendix A] for both the Trapezoidal rule and Simpson's rule. The former rule was shown to have a large aliasing lobe at $\Theta = \omega h = 2\pi$, while the latter rule had an additional large lobe at $\Theta = \pi$, due to the alternating character of the Simpson weights; see [10; (A-6) and (A-8)]. This leads us to anticipate that the linear approximation procedure developed here for Bessel transform (11) will be subject to aliasing near

$\Theta = 2\pi$, while the parabolic approximation will be degraded earlier, namely near $\Theta = \pi$. This will be borne out by the numerical examples to follow.

GRAPHICAL RESULTS

The Bessel transform numerical integration rule for the linear approximation to $g(x)$ is given by (23) or (26), while the rule for the parabolic approximation to $g(x)$ is given by (30) or (36). The exact transforms (38) and (40), and the absolute errors associated with these two rules, are depicted in figures 3 and 4 for the Rayleigh and Gaussian functions $g(x)$ of (37) and (39), respectively, with sampling increment $h = .1$. The ordinates in all figures are the logarithm to the base 10 of the corresponding results, while the abscissas are linear in ω or Θ. The upper limit, x_r, of integration in (2) or (11) is taken large enough to guarantee a negligible contribution (less than 1E-20) to the truncation error.

In figure 3, the error for the parabolic fits is initially lower (for small ω) than for the linear fits; however, the linear error decays rapidly with ω, and stays below the parabolic error for larger ω. Both absolute errors flatten out and are not increasing with ω, at least for this range of ω values. The maximum value of Θ is .8, as indicated in the figure.

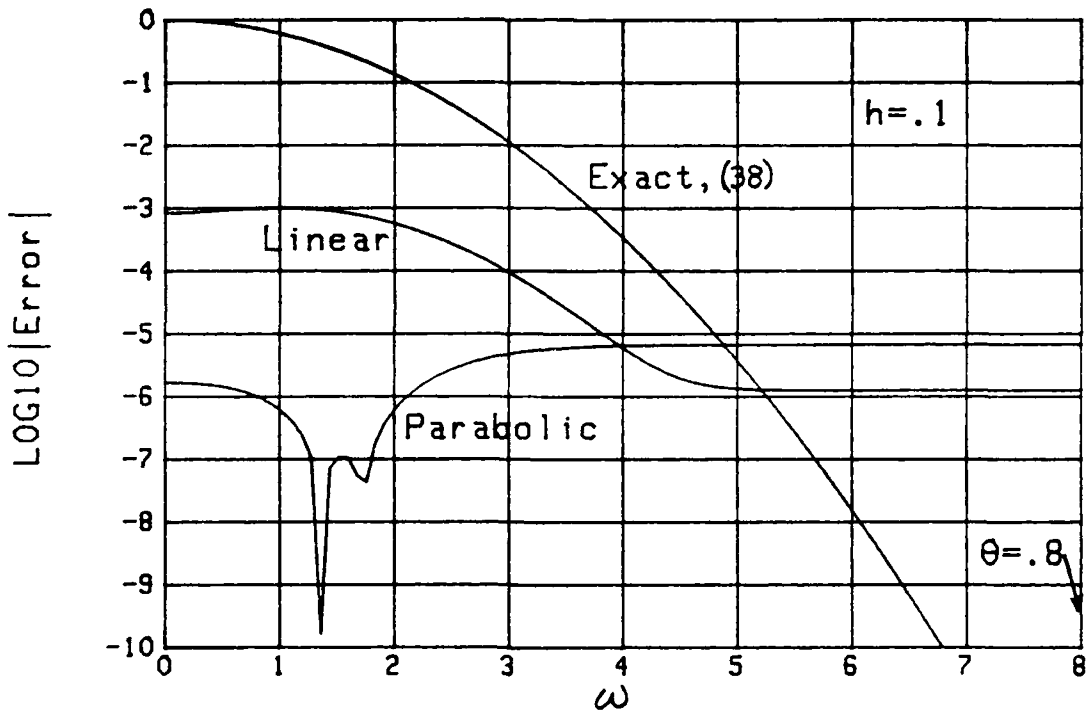

Figure 3. Errors for Rayleigh Function g(x)

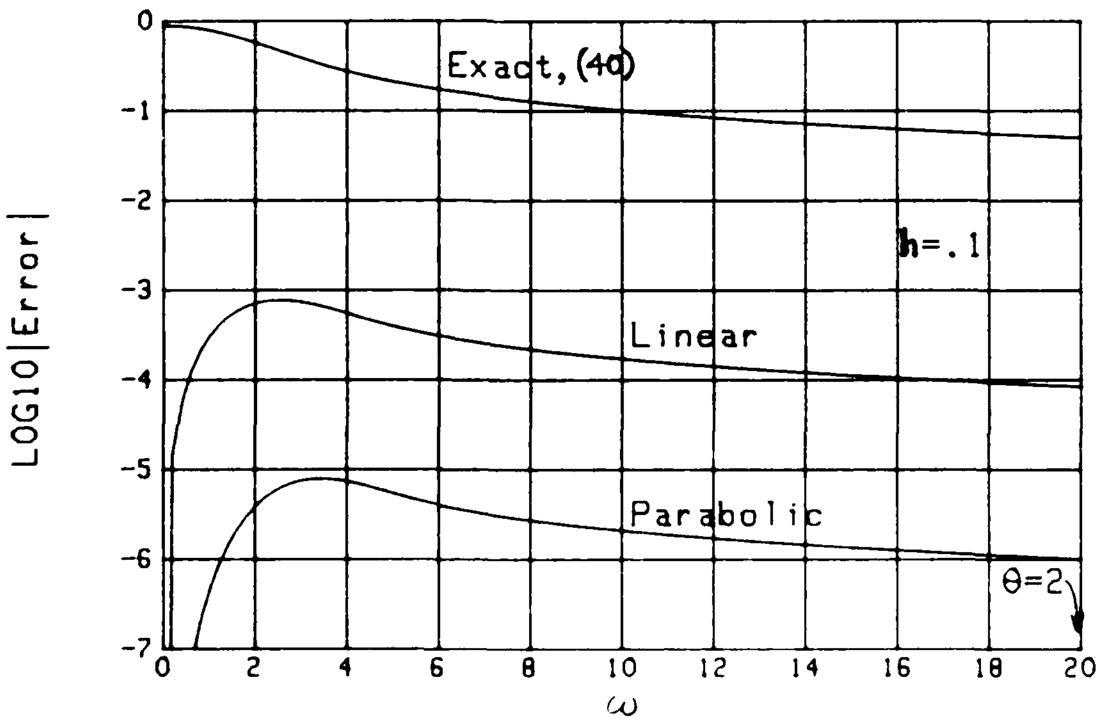

Figure 4. Errors for Gaussian Function g(x)

For the Gaussian function $g(x)$, the parabolic error in figure 4 is everywhere less than the linear error. Both errors near and at $\omega = 0$ are extremely small; this fortuitous result for the linear fits is fully explained in [6; pages 92-93], especially in the paragraph under (3.4.5). It has to do with the fact that the integrand in (11) for this Gaussian case, namely $J_0(\omega x) \exp(-x^2)$, has zero odd derivatives at the limits of integration. This is not the case for the Rayleigh function; hence the much larger errors at $\omega = 0$ in figure 3 result.

COMPARISON OF PROCEDURES

To demonstrate the benefits to be accrued from the fitting procedures derived in this study, a comparison of the absolute errors for four different procedures is presented in figure 5 for the Rayleigh function (37). The sampling increment in x is $h = .03$. The variable ω now covers the range (0,120); the point where $\Theta = \pi$ is indicated by a tic mark on the abscissa.

The Trapezoidal result is obtained by applying it to the complete integrand $J_0(\omega x)\, g(x)$ of (11). The error is essentially constant for all ω, including the region near $\Theta = \pi$; thus, as expected, aliasing is not significant at $\Theta = \pi$ for the Trapezoidal rule.

Application of the standard Simpson's rule to the complete integrand of (11) yields a very small error near $\omega = 0$, but a rapidly increasing error with ω, and a very large aliasing lobe centered around $\Theta = \pi$. This confirms the expectations presented earlier in this section.

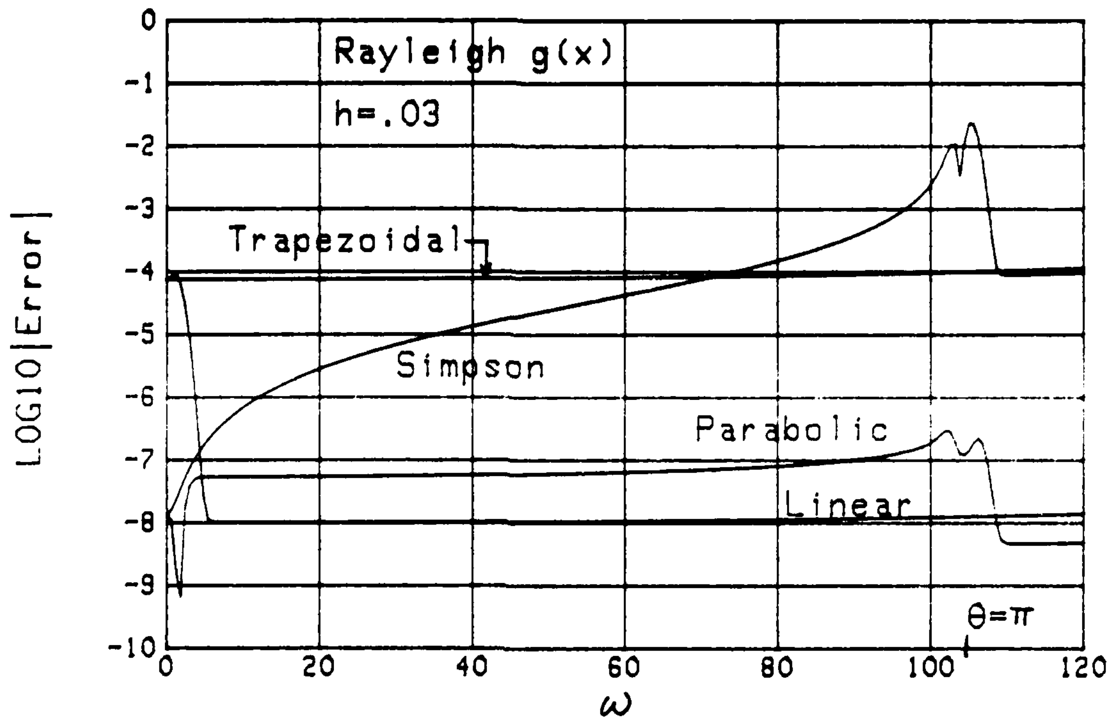

Figure 5. Errors for Four Procedures; ω<120

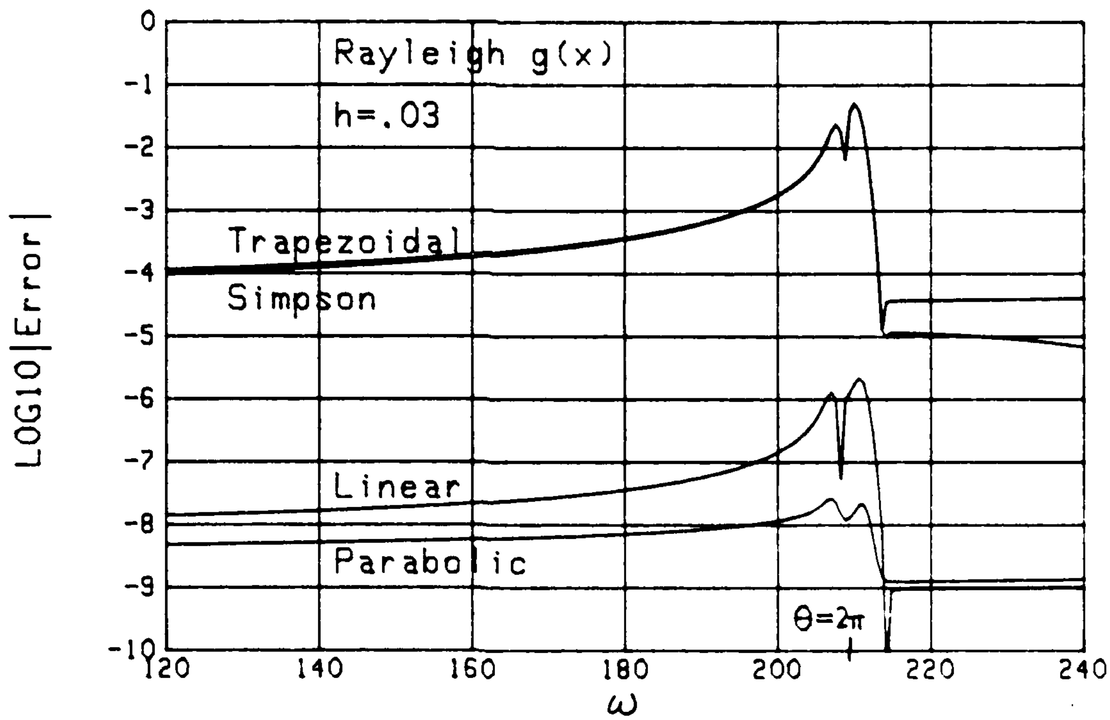

Figure 6. Errors for Four Procedures; ω>120

For the case of linear fits to $g(x)$, rather than $J_0(\omega x)\ g(x)$, the error drops dramatically, by four orders of magnitude as ω increases, similar to figure 3. Furthermore, there is no aliasing at $\Theta = \pi$.

The situation for the parabolic fits is that the absolute error starts out small and remains so, for all $\omega < 120$, there being a slight aliasing effect near $\Theta = \pi$. However, it is 5 orders of magnitude smaller than the Simpson error in this region of ω.

The results in figure 6 extend the abscissa to cover the range of (120,240) in ω; that is, these curves are an extension of those in figure 5. Now all rules suffer aliasing in the neighborhood of $\Theta = 2\pi$. The absolute error for the linear procedure increases by 2 orders of magnitude near $\Theta = 2\pi$, while the parabolic error is just slightly larger; however, the latter is 6 orders of magnitude better than the standard Trapezoidal and Simpson rules for numerical integration. All of these results confirm the predicted presence and location of aliasing discussed earlier.

ERROR DEPENDENCE ON SAMPLING INCREMENT

In figure 7, we investigate the dependence of the error on increment h employed to sample x in (11). Here we apply the linear fit procedure to the Rayleigh function (37). The absolute error for small ω (< 2) decreases by a factor of 4 as h is halved; that is, the large error bump near $\omega = 0$ behaves as h^2 for small increments h. On the other hand, for larger ω (> 5),

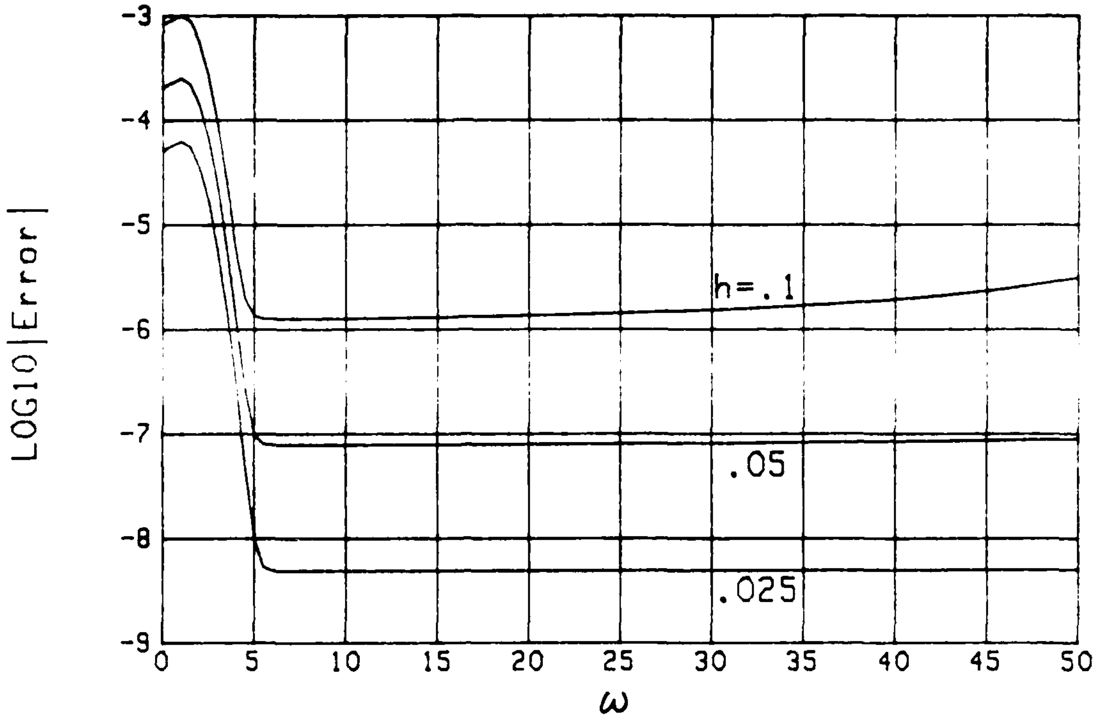

Figure 7. Linear Procedure, Rayleigh g(x)

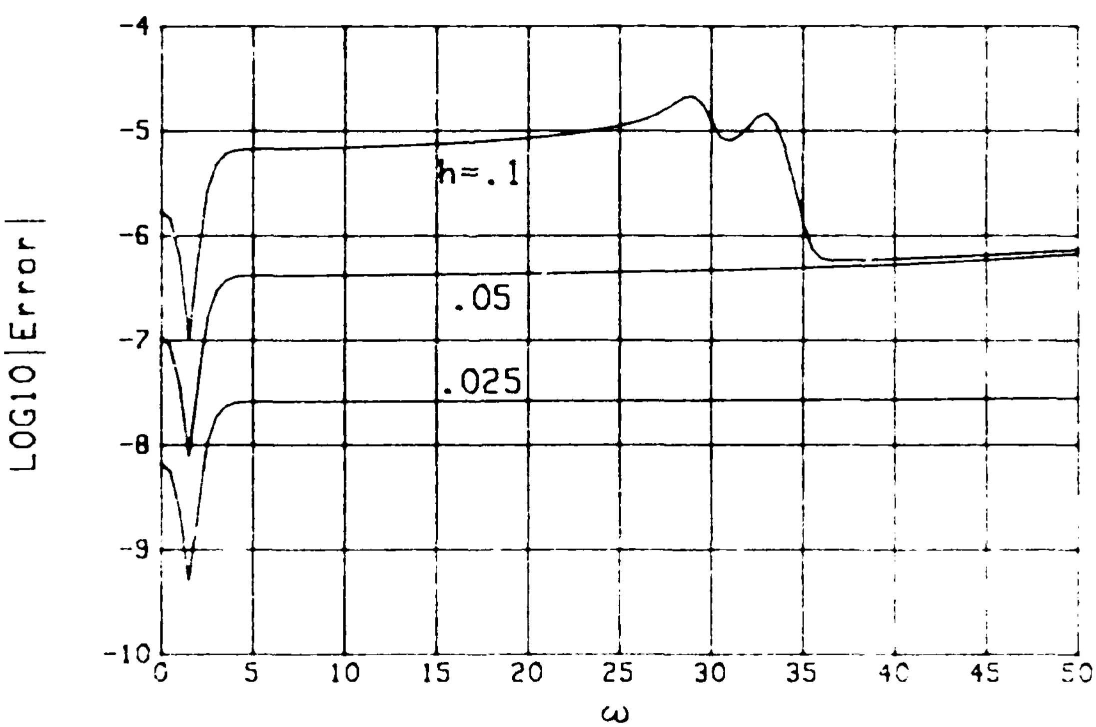

Figure 8. Parabolic Procedure, Rayleigh g(x)

the error decreases by a factor of 16 when h is halved; that is, the "saturation" level of error behaves as h^4 for small h. The slight flare in the error curve near $\omega = 50$, for $h = .1$, is an indication of the beginning of aliasing; that is, $\Theta = 5$ here, which is near the $\Theta = 2\pi$ location.

Still considering the Rayleigh function (37), but now switching to the parabolic procedure, the results in figure 8 demonstrate that the error drops by a factor of 16 as h is halved; thus, the error dependence is h^4 for all ω. The wiggles in the $h = .1$ curve near $\omega = 30$ are due to aliasing, since $\Theta = \pi$ for $\omega = 10\pi = 31.4$.

When the function $g(x)$ is changed to the Gaussian example of (39), and the linear fitting procedure is employed, the errors are depicted in figure 9. Here, the error dependence is according to h^2 for all ω, until aliasing sets in. Aliasing is present in the $h = .1$ curve near $\omega = 64$, since $\Theta = 2\pi$ at $\omega = 62.8$ for that curve. Comparison of these errors with the exact answer in figure 4 reveals that the relative error is constant in the range $4 < \omega < 56$.

When the parabolic procedure is used instead on the Gaussian example, the error dependence is again according to h^4, until aliasing becomes dominant. The aliasing lobes in the $h = .1$ curve in figure 10 are centered at $\Theta = \pi$ and 2π, as before. The large increase in the error for the $h = .025$ curve, when ω exceeds 50, is a feature not seen previously. It may be due to the sum of distant aliasing of sidelobes which decay very slowly

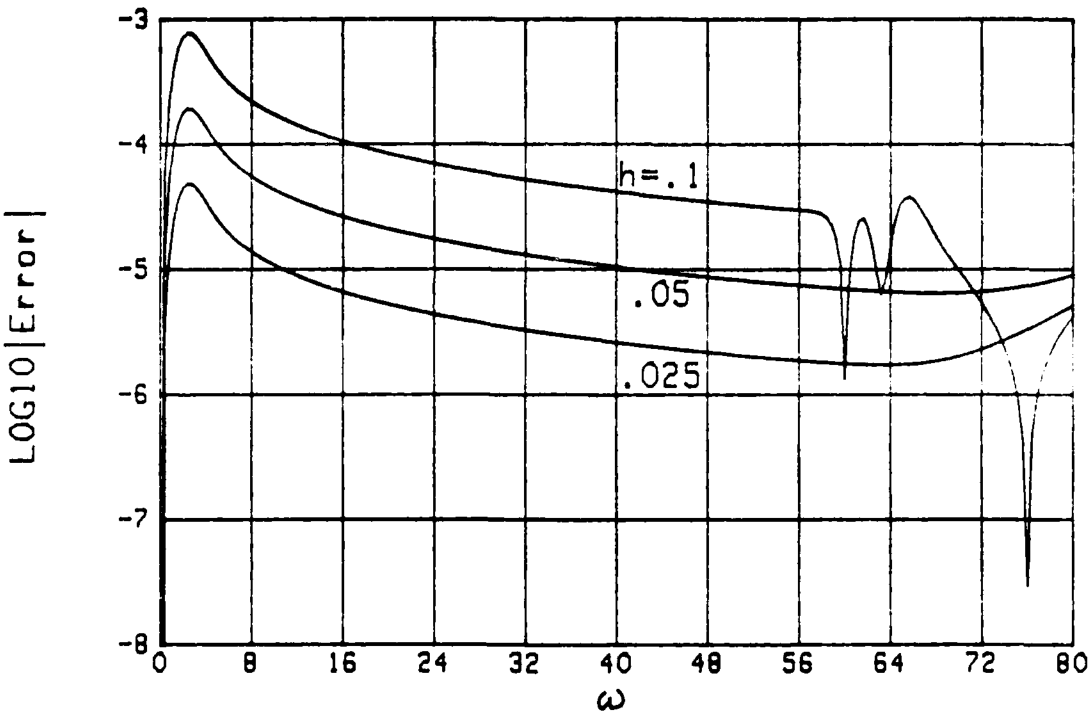

Figure 9. Linear Procedure, Gaussian g(x)

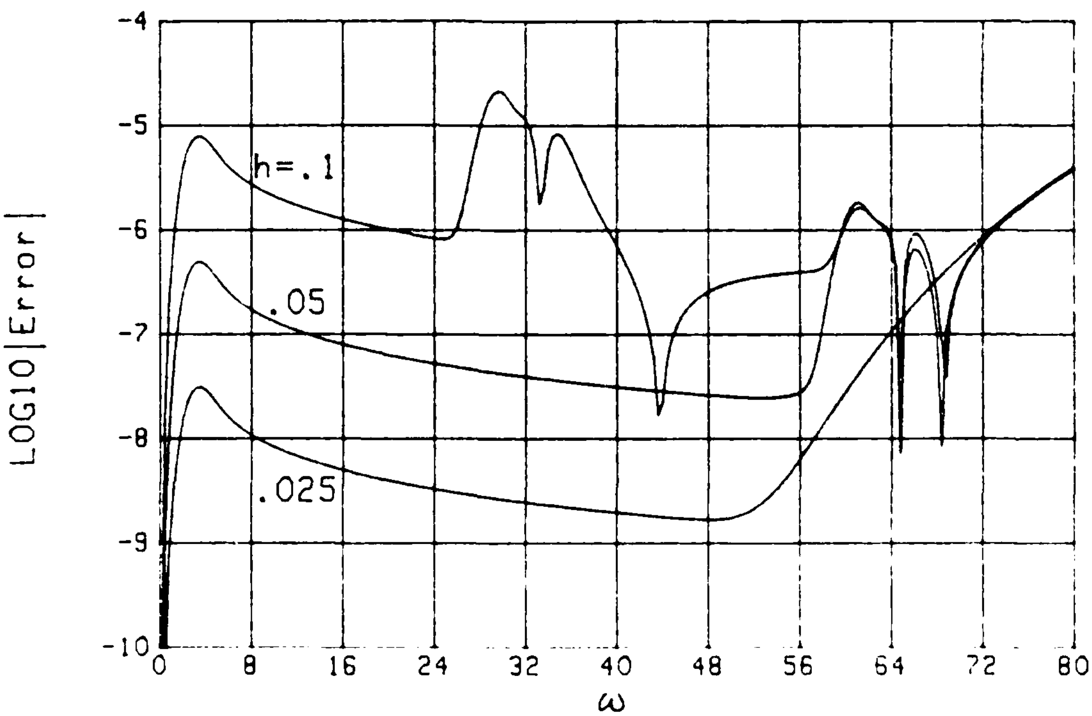

Figure 10. Parabolic Procedure, Gaussian g(x)

with ω; in fact, from (41), the exact answer only decays as $1/\omega$. The rapid decay of the Rayleigh transform, (38), apparently precluded this type of error from appearing in any of the numerical cases considered here for the Rayleigh $g(x)$.

SUMMARY

There is a marked difference between the form of these results and the Filon equations; namely, the term multiplying sample value $g_n = g(x_n)$ (in (B-3), for example) varies with n in such a fashion that no simplification or factoring is possible. In order to better explain this complication, let us investigate the evaluation of (18) when $J_o(\omega x)$ is replaced by $\exp(i\omega x)$; that is, consider evaluation of a Fourier transform, rather than a Bessel transform, for the moment. When the linear fits to g(x) in (18) are then integrated, there follows

$$I_n = g_n \; h \; \exp(in\theta) \left[\frac{\sin(\theta/2)}{\theta/2}\right]^2 . \qquad (44)$$

But the bracketed term here is a common factor (independent of n) that can be removed from the summation on n. This fortuitous simplification does not hold for the corresponding result (B-3) here, because whereas exp(iu) is periodic, $J_o(u)$ and A(u) are not.

In an effort to recover some of this loss in execution time, we therefore grouped the terms in (B-6) in an alternative form, pivoted around $B_1(n\theta)$ rather than g_n; see (B-7). Perhaps another rearrangement of terms would be more advantageous for some purposes.

It is possible to extend the results here to other Bessel transforms. For example, suppose we are interested in the evaluation of first-order transform

$$\int dx \ J_1(\omega x) \ g(x) \ , \tag{45}$$

and we approximate g(x) either by straight lines or parabolas. The integrals in (13)-(15) are then replaced by

$$\int_0^u dt \ J_1(t) = 1 - J_0(u) \ ,$$

$$\int_0^u dt \ t \ J_1(t) = B_0(u) \ ,$$

$$\int_0^u dt \ t^2 \ J_1(t) = u^2 \ J_2(u) = 2u \ J_1(u) - u^2 \ J_0(u) \ , \tag{46}$$

where we used [5; (11.1.6) and (9.1.30)] and (16). Since all of these terms have already been encountered here, extension to transform (45) would not be difficult.

For the evaluation of the alternative transform

$$\int dx \ \frac{J_1(\omega x)}{x} \ g(x) \ , \tag{47}$$

we need the additional result [5; (11.1.1)]

$$\int_0^u dt \, \frac{J_1(t)}{t} = \frac{4}{u} \sum_{k=1}^{\infty} k \, J_{2k}(u) =$$

$$= \frac{4}{u}\left[J_2(u) + 2 \, J_4(u) + 3 \, J_6(u) + \ldots\right]. \tag{48}$$

But this type of term is easily evaluated by means of the downward recurrence technique given in appendix A. In fact, immediately following the single line Se = Se + E, we have merely to add the line Sx = Sx + Se; when the downward recurrence is completed, the bracketed term in (48) results in storage location Sx (after the scaling correction).

APPENDIX A

NUMERICAL EVALUATION PROCEDURE FOR BESSEL INTEGRALS

The three fundamental Bessel integrals that must be evaluated are given by (13)-(17) as

$$A(u) = \int_0^u dt\ J_o(t)\ , \qquad \text{(A-1)}$$

$$B_o(u) = A(u) - u\ J_o(u) = \int_0^u dt\ t\ (u - t)\ J_o(t)\ , \qquad \text{(A-2)}$$

$$B_1(u) = A(u) - J_1(u) = \int_0^u dt \left(1 - \frac{t}{u}\right) J_o(t)\ . \qquad \text{(A-3)}$$

By expanding J_o in a power series [5; (9.1.10)], and integrating term by term, there follows from (A-1),

$$A(u) = \frac{u}{2} \sum_{k=0}^{\infty} \frac{(-u^2/4)^k}{k!\ k!\ (k+\frac{1}{2})} = u - \frac{u^3}{12} + \frac{u^5}{320} - \dots\ . \qquad \text{(A-4)}$$

When this result is coupled with the series expansions of J_o and J_1 in (A-2) and (A-3) respectively, there follows

$$B_o(u) = \frac{u^3}{4} \sum_{k=0}^{\infty} \frac{(-u^2/4\;)^k}{k!\;(k+1)!\;(k+\frac{3}{2})} = \frac{u^3}{6} - \frac{u^5}{80} + \frac{u^7}{2688} - \ldots \qquad \text{(A-5)}$$

and

$$B_1(u) = \frac{u}{4} \sum_{k=0}^{\infty} \frac{(-u^2/4\;)^k}{k!\;(k+1)!\;(k+\frac{1}{2})} = \frac{u}{2} - \frac{u^3}{48} + \frac{u^5}{1920} - \ldots\;. \qquad \text{(A-6)}$$

Although these power series could be used for small and moderate values of u, they are not useful for large u, due to the loss of significant digits caused by the alternating character of series (A-4)-(A-6). In fact, we will find that a downward recurrence will yield all the values of A, B_o, B_1, J_o, and J_1 very efficiently for small u, while an asymptotic expansion is equally attractive for large u.

DOWNWARD RECURRENCE

We start with [5; (11.1.2)] and (A-1):

$$A(u) = 2[J_1(u) + J_3(u) + J_5(u) + \ldots]\;. \qquad \text{(A-7)}$$

Thus if we can evaluate all the odd-order Bessel functions, we can get A(u) from their sum. Also, $B_o(u)$ and $B_1(u)$ follow immediately from (A-2) and (A-3), if we can additionally get $J_o(u)$.

But the Bessel functions satisfy the downward recurrence [5; (9.1.27)], [illegible]

$$J_m(u) = \frac{2}{u}\,(m + 1)\; J_{m+1}(u) - J_{m+2}(u) \tag{A-8}$$

for $m \geq 0$. This recurrence can be started by guessing at $J_M(u) = 0$, $J_{M-1}(u) = 1E{-}250$ for example, and evaluating downward via (A-8) to $m = 0$. Since the error increases much slower than the size of the terms in (A-8) [5; table 9.4], the relative error of the terms is very small for the smaller values of m, if M is chosen large enough to start with. In order to accurately establish the absolute level of the sequence of $\{J_m\}$ values, we then use the check sum formula [5; (9.1.46)]

$$J_0(u) + 2[J_2(u) + J_4(u) + \ldots] = 1 \;. \tag{A-9}$$

In order to realize 15 decimal accuracy in A, B_0, B_1, J_0, J_1, it has been found sufficient to choose even integer M as

$$M = M(u) = 2\ \mathrm{INT}\left(20 + .56u - \frac{175}{12 + u}\right) + 12 \quad \text{for } 0 \leq u < 45 \;. \tag{A-10}$$

While conducting the downward recurrence on m in (A-8), an even sum of $J_M + J_{M-2} + \ldots$, and an odd sum of $J_{M-1} + J_{M-3} + \ldots$, are maintained. After completion to $m = 0$, the even sum is subject to constraint (A-9), in order to establish the scale factor that must be applied to all the desired outputs; this is to correct for the initial arbitrary (incorrect) guess of $J_{M-1}(u) = 1E{-}250$. With this scale factor in hand, the odd sum in (A-7) can then be modified by means of one multiplication for the correct absolute level for A(u). Since the last two quantities yielded by recurrence (A-8)

are $J_1(u)$ and $J_0(u)$ (after scaling), we then have all the necessary ingredients to determine $B_0(u)$ and $B_1(u)$.

No array declarations or array storage is necessary in this procedure, since there is never any need to "go back up" the recurrence and correctly scale the $\{J_m(u)\}$ terms. This has been guaranteed (through numerical investigation) by the choice of M in (A-10). A further economy in the program for this two-term recurrence (A-8) has been achieved by splitting it into even and odd versions, thereby avoiding the usual temporary storage of the left-hand side of (A-8) until the right-hand side is updated. This compact program is listed below as subroutine SUB Besj. For given u, it outputs values for $J_0(u)$, $J_1(u)$, $A(u)$, $B_0(u)$, $B_1(u)$, provided that $0 \le u < 45$.

ASYMPTOTIC EXPANSION

For large u, the starting integer M in (A-10) gets too large to make downward recurrence a viable procedure. Instead, we resort to the asymptotic expansion [5; (11.1.11)]

$$A(u) = \int_0^u dt\, J_0(t) \sim 1 -$$

$$- \left(\frac{2}{\pi u}\right)^{1/2} \left[\cos\left(u - \frac{\pi}{4}\right) \sum_{k=0}^{\infty} \frac{(-1)^k\, a_{2k+1}}{u^{2k+1}} - \sin\left(u - \frac{\pi}{4}\right) \sum_{k=0}^{\infty} \frac{(-1)^k\, a_{2k}}{u^{2k}}\right] \qquad \text{(A-11)}$$

as $u \to +\infty$; here, we also used the definite integral result that $A(\infty) = 1$ [5; (11.4.17)]. The values of the coefficients are [5; (11.1.2)]

$$a_k = \left(\frac{1}{2}\right)_k \sum_{s=0}^{k} \left(\frac{1}{2}\right)_s \frac{1}{2^s s!} \tag{A-12}$$

and are conveniently obtained by recursion

$$T_s \equiv \left(\frac{1}{2}\right)_s \frac{1}{2^s s!} = \frac{s - \frac{1}{2}}{2s} T_{s-1} \quad \text{for } s \geq 1 . \tag{A-13}$$

The number of terms required in the summations in (A-11) depends on the value of u and the desired accuracy. For $u > 45$ and 15 decimal accuracy, it has been sufficient to terminate (A-11) at $k = \mathrm{INT}(u/2)$.

Since (A-11) yields only $A(u)$, it is necessary to calculate $J_0(u)$ and $J_1(u)$ additionally; this has been accomplished by use of [11; section 6.8]. All of these quantities are evaluated by means of subroutine SUB Bessel listed below. For input $u \geq 0$, this subroutine yields values of $J_0(u)$, $J_1(u)$, $A(u)$, $B_0(u)$, $B_1(u)$.

```
10    SUB Bessel(X,J0,J1,A,B0,B1)       !  A = INTEGRAL(0,X) dt Jo(t)
20    DOUBLE K,I                        !  INTEGERS
30    IF X>45. THEN 60
40    CALL Besj(X,J0,J1,A,B0,B1)        !  DOWNWARD RECURRENCE 9.1.27,1
50    SUBEXIT
60    I=INT(X)/2                        !  ASYMPTOTIC SERIES 11.1.11 & 12
70    Rx=1./X
80    F=.5*Rx
90    T=.25
100   A=1.25
110   Re=.625*Rx
120   Im=P=1.
130   FOR K=1 TO I
140   P=-P
150   Sn=K+K
160   F5=Sn-.5
170   F=F*F5*Rx
180   T=T*F5/(Sn+Sn)
190   A=A+T
200   Be=F*A
210   Im=Im+P*Be
220   Sn=Sn+1.
230   F5=Sn-.5
240   F=F*F5*Rx
250   T=T*F5/(Sn+Sn)
260   A=A+T
270   Bo=F*A
280   Re=Re+P*Bo
290   IF Be*Be+Bo*Bo<1.E-26 THEN 310
300   NEXT K
310   F=X-.785398163397448 28
320   T=.797884560802865 41
330   A=1.-T*SQR(Rx)*(Re*COS(F)-Im*SIN(F))
340   J0=FNJo(X)                        !  J0 = Jo(X)
350   J1=FNJ1(X)                        !  J1 = J1(X)
360   B0=A-X*J0                         !  B0 = A(X) - X Jo(X)
370   B1=A-J1                           !  B1 = A(X) - J1(X)
380   SUBEND
390   !
400   SUB Besj(U,J0,J1,A,B0,B1)         !  J0 = Jo(U),  J1 = J1(U)
410   IF U>0. THEN 450                  !   A = A(U) = INTEGRAL(0,U) dt Jo(t)
420   J0=1.                             !  B0 = A(U) - U Jo(U)
430   J1=A=B0=B1=0.                     !  B1 = A(U) - J1(U)
440   SUBEXIT
450   DOUBLE Mc,Ms                      !  INTEGERS
460   Mc=2*INT(20.+.56*U-175./(12.+U))+12
470   T=2./U
480   Se=E=0.
490   So=O=1.E-250
500   FOR Ms=Mc TO 2 STEP -2
510   E=T*(Ms+1)*O-E                    !  9.1.27,1
520   Se=Se+E
530   O=T*Ms*E-O                        !  9.1.27,1
540   So=So+O
550   NEXT Ms
560   E=T*O-E
570   F=1./(Se+Se+E)                    !  9.1.46
580   J0=E*F
590   J1=O*F
600   A=(So+So)*F                       !  11.1.2
610   B0=A-U*J0
620   B1=A-J1
630   SUBEND
640   !
```

```
650   DEF FNJo(X)          !  Jo(X) via Hart #5845, 6546, and 6946
660   Y=ABS(X)
670   IF Y>8. THEN 770
680   T=Y*Y
690   P=2271490439.5536033-T*(5513584.5647707522-T*5292.6171303845574)
700   P=2334489171877869.7-T*(4776555944267 3.588-T*(462172225031.71803-T*P))
710   P=18596231762189780 4.-T*(44145829391815982.-T*P)
720   Q=204251483.52134357+T*(494030.79491813972+T*(884.72036756175504+T))
730   Q=2344750013658996.8+T*(15015462449769.752+T*(64398674535.133256+T*Q))
740   Q=18596231762189773 3.+T*Q
750   Jo=P/Q
760   RETURN Jo
770   Z=8./Y
780   T=Z*Z
790   Pn=2204.5010439651804+T*(128.67758574871419+T*.90047934748028803)
800   Pn=8554.8225415066617+T*(8894.4375329606194+T*Pn)
810   Pd=2214.0488519147104+T*(130.88490049992388+T)
820   Pd=8554.8225415066628+T*(8903.8361417095954+T*Pd)
830   Qn=13.990976865960680+T*(1.0497327982345548+T*.0093525953294 0319)
840   Qn=37.510534954957112+T*(46.093826814625175+T*Qn)
850   Qd=921.56697552653090+T*(74.428389741411179+T)
860   Qd=2400.6742371172675+T*(2971.9837452084920+T*Qd)
870   T=Y-.78539816339744828
880   Jo=.28209479177387820*SQR(Z)*(COS(T)*Pn/Pd+SIN(T)*Z*Qn/Qd)
890   RETURN Jo
900   FNEND
910   !
920   DEF FNJ1(X)          !  J1(X) via Hart #6045, 6747, and 7147
930   Y=ABS(X)
940   IF Y>8. THEN 1040
950   T=Y*Y
960   P=.11073522244537306E-10-T*.63194310317443161E-14
970   P=.49105992765551294E-5-T*(.93821933651407445E-8-T*P)
980   P=.39831079839523 3-T*(.17057692643496171E-2-T*P)
990   P=5878.7877666568200-T*(61.218769973569439-T*P)
1000  P=69536422.632983850-T*(8356785.4873489143-T*(320302.74688539470-T*P))
1010  Q=139072845.26596769+T*(670534.68354822993+T*(1284.5934539663019+T))
1020  J1=X*P/Q
1030  RETURN J1
1040  Z=8./Y
1050  T=Z*Z
1060  Pn=3132.7529563550695+T*(174.31379748379025+T*1.2265053764359043)
1070  Pn=12909.184718961881+T*(13090.420511035065+T*Pn)
1080  Pd=3109.2814167700288+T*(169.04721775008610+T)
1090  Pd=12909.184718961879+T*(13066.783087844020+T*Pd)
1100  Qn=51.736532818365916+T*(3.7994453796980673+T*.036363466476034711)
1110  Qn=144.65282874995209+T*(174.42916890924259+T*Qn)
1120  Qd=1119.1098527047487+T*(85.223920643413404+T)
1130  Qd=3065.9270133323172+T*(3734.3401060163013+T*Qd)
1140  T=Y-2.3561944901923448
1150  J1=.28209479177387820*SQR(Z)*(COS(T)*Pn/Pd-SIN(T)*Z*Qn/Qd)
1160  IF X<0. THEN J1=-J1
1170  RETURN J1
1180  FNEND
```

APPENDIX B

DERIVATION OF INTEGRATION RULE FOR STRAIGHT LINE FITS TO g(x)

The situation of interest here is represented in figure 1, where straight lines are fit to g(x) between adjacent samples of g(x), taken at sample points $\{x_n\}$. In particular, the contribution to integral (11) of an (internal) abutting point x_n was set up in (18)-(19). By letting $t = \omega x$ in (18), and using (19), (20), and (22), namely

$$y = \frac{x - x_n}{h} \;, \; x_n = nh \;, \; \Theta = \omega h \;, \tag{B-1}$$

there follows, for the n-th contribution to the integral,

$$I_n = \frac{g_n}{\omega} \int_{(n-1)\Theta}^{n\Theta} dt \; J_0(t) \left(1 - n + \frac{t}{\Theta}\right) +$$

$$+ \frac{g_n}{\omega} \int_{n\Theta}^{(n+1)\Theta} dt \; J_0(t) \left(1 + n - \frac{t}{\Theta}\right) , \tag{B-2}$$

where $g_n = g(x_n) = g(nh)$. By reference to the auxiliary functions defined in (13)-(17), the sum of these two integrals can be expressed in the compact form

$$I_n = \frac{g_n}{\omega}\Big\{(n + 1) \; B_1[(n + 1)\Theta] - 2n \; B_1[n\Theta] + (n - 1) \; B_1[(n - 1)\Theta]\Big\}$$
$$\text{for } \ell < n < r \; . \tag{B-3}$$

The procedures in appendix A are now directly applicable to the evaluation of (B-3) for any n.

For the left-end point x_ℓ depicted on the left side of figure 1, the corresponding contribution to desired integral result (11) is, using (B-1) again,

$$I_\ell = \int_{x_\ell}^{x_\ell + h} dx\, J_0(\omega x)\, g_\ell\, (1 - y) =$$

$$= \frac{g_\ell}{\omega} \int_{\ell\Theta}^{(\ell+1)\Theta} dt\, J_0(t) \left(\ell + 1 - \frac{t}{\Theta}\right) =$$

$$= \frac{g_\ell}{\omega}\left\{(\ell + 1)\, B_1\left[(\ell + 1)\Theta\right] - (\ell + 1)\, B_1\left[\ell\Theta\right] - J_1\left[\ell\Theta\right]\right\} . \qquad \text{(B-4)}$$

The corresponding contribution to integral (11) for the right-end point x_r is given by

$$I_r = \int_{x_r - h}^{x_r} dx\, J_0(\omega x)\, g_r\, (1 + y) =$$

$$= \frac{g_r}{\omega} \int_{(r-1)\Theta}^{r\Theta} dt\, J_0(t) \left(\frac{t}{\Theta} - r + 1\right) =$$

$$= \frac{g_r}{\omega}\left\{(r - 1)\, B_1\left[(r - 1)\Theta\right] - (r - 1)\, B_1\left[r\Theta\right] + J_1\left[r\Theta\right]\right\} . \qquad \text{(B-5)}$$

(As a check, combination of (B-4) and (B-5), upon replacement of ℓ and r by n, yields (B-3), as it should. The "end correction terms" in J_1 cancel out for all internal points, n.)

The resultant approximation to desired integral (11) is given by the sum of (B-3)-(B-5):

$$G(\omega) = \int_{x_\ell}^{x_r} dx\ J_0(\omega x)\ g(x) \cong I_\ell + I_r + \sum_{n=\ell+1}^{r-1} I_n\ . \tag{B-6}$$

This particular grouping of terms is according to the function sample values $\{g_n\} = \{g(nh)\}$. An alternative grouping, according to the samples of function $B_1(u)$ instead, is given by

$$\begin{aligned} \omega\ G(\omega) \cong & \left[\ell\ g_{\ell+1} - (\ell + 1)g_\ell\right] B_1(\ell\Theta) - g_\ell\ J_1(\ell\Theta) + \\ & + \left[r\ g_{r-1} - (r - 1)g_r\right] B_1(r\Theta) + g_r\ J_1(r\Theta) + \\ & + \sum_{n=\ell+1}^{r-1} n\left[g_{n+1} - 2g_n + g_{n-1}\right] B_1(n\Theta)\ . \end{aligned} \tag{B-7}$$

Whereas $B_1(n\Theta)$ must be evaluated for all $\ell \le n \le r$, the J_1 function need only be evaluated at the end points $\ell\Theta$ and $r\Theta$.

When ω is restricted to be multiples of a sampling increment Δ, that is

$$\omega = k\Delta \quad \text{for } k = 1, 2, \ldots \tag{B-8}$$

then (B-7) yields, for $k \geq 1$, the approximation

$$\begin{aligned} k\Delta\, G(k\Delta) \cong & \left[\ell\, g_{\ell+1} - (\ell + 1)g_{\ell}\right] B_1(\ell k\Delta h) - g_{\ell}\, J_1(\ell k\Delta h) + \\ & + \left[r\, g_{r-1} - (r - 1)g_r\right] B_1(rk\Delta h) + g_r\, J_1(rk\Delta h) + \\ & + \sum_{n=\ell+1}^{r-1} n\left[g_{n+1} - 2g_n + g_{n-1}\right] B_1(nk\Delta h) \,, \end{aligned} \tag{B-9}$$

where

$$g_n = g(nh) \,. \tag{B-10}$$

Since n and k are integers (see (20) and (B-8)), the evaluation of $B_1(u)$ in (B-9) is confined to integer multiples of Δh, i.e. $u = m\Delta h$. Further discussion on how to take advantage of this feature of (B-9) is given in the sequel to (26). The end result is that we have two alternative procedures for evaluation of (B-9) and two corresponding programs: one faster routine which may require considerable storage, and a slower procedure utilizing very little storage. Programs for both procedures are listed below.

BEHAVIOR FOR SMALL Θ

When Θ is small, the differences of like quantities in (B-3)-(B-5) can be circumvented by expanding B_1 and J_1 in power series in Θ. Using the facts that

$$B_1(u) \sim \frac{u}{2} - \frac{u^3}{48} \quad \text{as } u \to 0 \ ,$$

$$J_1(u) \sim \frac{u}{2} - \frac{u^3}{16} \quad \text{as } u \to 0 \ , \qquad \text{(B-11)}$$

the above results reduce to

$$I_n \sim \quad g_n \, h\left[1 - \frac{1}{4}\,\Theta^2\left(n^2 + \frac{1}{6}\right)\right] ,$$

$$I_\ell \sim \frac{1}{2}\, g_\ell \, h\left[1 - \frac{1}{4}\,\Theta^2\left(\ell^2 + \frac{2}{3}\ell + \frac{1}{6}\right)\right] ,$$

$$I_r \sim \frac{1}{2}\, g_r \, h\left[1 - \frac{1}{4}\,\Theta^2\left(r^2 - \frac{2}{3}\, r + \frac{1}{6}\right)\right] , \qquad \text{(B-12)}$$

as $\Theta \to 0$. By use of the power series expansion developed for $B_1(u)$ in appendix A, these results could be extended to order Θ^4, Θ^6 if desired.

The total contribution to (11) is given by the sum in (B-6). As $\Theta \to 0$, this reduces to the Trapezoidal rule, (12).

```
 10   ! ZERO-TH ORDER BESSEL TRANSFORM USING LINEAR INTERPOLATION.
 20   ! INTEGRAL(Xl,Xr) dX Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
 30   ! Gw(Ks), where W = Ks*Delw.        Faster high-storage.
 40     Delx=.025                    !   INCREMENT (h) IN X
 50     L=0                          !   Xl=L*Delx, L>=0
 60     R=400                        !   Xr=R*Delx, R>L
 70     Delw=.2                      !   INCREMENT (▲) IN W
 80     K1=0                         !   W1=K1*Delw, K1>=0
 90     K2=40                        !   W2=K2*Delw, K2>=K1
100     DOUBLE L,R,K1,K2,K0,L1,R1,Ns,Ks,I       !  INTEGERS
110     DIM Gx(500),Dg(500),B1(50000),J1l(100),J1r(100),Gw(100)
120     K0=K1
130     K1=MAX(K1,1)
140     L1=L+1
150     R1=R-1
160     REDIM Gx(L:R),Dg(L1:R),B1(L*K1:R*K2)
170     REDIM J1l(K1:K2),J1r(K1:K2),Gw(K0:K2)
180     FOR Ks=K0 TO K2
190     Gw(Ks)=0.
200     NEXT Ks
210     FOR Ns=L TO R
220     Gx(Ns)=FNG(Ns*Delx)          !   SEE DEF FNG(X) = g(X)
230     NEXT Ns
240     Gl=Gx(L)
250     Gr=Gx(R)
260     IF K0>0 THEN 320
270     F=.5*(Gl+Gr)
280     FOR Ns=L1 TO R1
290     F=F+Gx(Ns)
300     NEXT Ns
310     Gw(0)=F*Delx
320     FOR Ns=L1 TO R
330     Dg(Ns)=Gx(Ns)-Gx(Ns-1)
340     NEXT Ns
350     D2=Delw*Delx
360     IF L=0 THEN 410
370     FOR Ks=K1 TO K2
380     I=L*Ks
390     CALL Bessel(I*D2,J0,J1l(Ks),A,B0,B1(I))
400     NEXT Ks
410     FOR Ks=K1 TO K2
420     I=R*Ks
430     CALL Bessel(I*D2,J0,J1r(Ks),A,B0,B1(I))
440     NEXT Ks
450     FOR Ns=L1 TO R1
460     FOR Ks=K1 TO K2
470     I=Ns*Ks
480     IF B1(I)<>0. THEN 500
490     CALL Bessel(I*D2,J0,J1,A,B0,B1(I))
500     NEXT Ks
510     NEXT Ns
```

```
520     T1=L*Dg(L1)-Gl
530     T2=R*Dg(R)-Gr
540     IF L=0 THEN 580
550     FOR Ks=K1 TO K2
560     Gw(Ks)=T1*B1(L*Ks)-Gl*J1l(Ks)
570     NEXT Ks
580     FOR Ks=K1 TO K2
590     F=T2*B1(R*Ks)-Gr*J1r(Ks)
600     Gw(Ks)=Gw(Ks)-F
610     NEXT Ks
620     FOR Ns=L1 TO R1
630     F=Ns*(Dg(Ns+1)-Dg(Ns))
640     FOR Ks=K1 TO K2
650     Gw(Ks)=Gw(Ks)+F*B1(Ns*Ks)
660     NEXT Ks
670     NEXT Ns
680     FOR Ks=K1 TO K2
690     Gw(Ks)=Gw(Ks)/(Ks*Delw)
700     NEXT Ks
710     PRINT Gw(*)
720     PAUSE
730     END
740     !
750     DEF FNG(X)                       !   g(X)
760       Gx=X*EXP(-.5*X*X)              !   RAYLEIGH EXAMPLE
770     RETURN Gx
780     FNEND
```

```
 10  ! ZERO-TH ORDER BESSEL TRANSFORM USING LINEAR INTERPOLATION.
 20  ! INTEGRAL(X1,Xr) dX Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
 30  ! Gw(Ks), where W = Ks*Delw.       Slower low-storage.
 40    Delx=.025                      !  INCREMENT (h) IN X
 50    L=0                            !  X1=L*Delx, L>=0
 60    R=400                          !  Xr=R*Delx, R>L
 70    Delw=.5                        !  INCREMENT (Δ) IN W
 80    K1=0                           !  W1=K1*Delw, K1>=0
 90    K2=100                         !  W2=K2*Delw, K2>=K1
100    DOUBLE L,R,K1,K2,K0,L1,R1,Ns,Ks         !  INTEGERS
110    DIM Gx(500),Dg(500),Gw(200)
120    K0=K1
130    K1=MAX(K1,1)
140    L1=L+1
150    R1=R-1
160    REDIM Gx(L:R),Dg(L1:R),Gw(K0:K2)
170    FOR Ks=K0 TO K2
180    Gw(Ks)=0.
190    NEXT Ks
200    FOR Ns=L TO R
210    Gx(Ns)=FNG(Ns*Delx)            !  SEE DEF FNG(X) = g(X)
220    NEXT Ns
230    Gl=Gx(L)
240    Gr=Gx(R)
250    IF K0>0 THEN 310
260    F=.5*(Gl+Gr)
270    FOR Ns=L1 TO R1
280    F=F+Gx(Ns)
290    NEXT Ns
300    Gw(0)=F*Delx
310    FOR Ns=L1 TO R
320    Dg(Ns)=Gx(Ns)-Gx(Ns-1)
330    NEXT Ns
340    D2=Delw*Delx
350    T1=L*Dg(L1)-Gl
360    T2=R*Dg(R)-Gr
370    IF L=0 THEN 430
380    T=L*D2
390    FOR Ks=K1 TO K2
400    CALL Bessel(T*Ks,J0,J1,A,B0,B1)
410    Gw(Ks)=T1*B1-Gl*J1
420    NEXT Ks
430    T=R*D2
440    FOR Ks=K1 TO K2
450    CALL Bessel(T*Ks,J0,J1,A,B0,B1)
460    F=T2*B1-Gr*J1
470    Gw(Ks)=Gw(Ks)-F
480    NEXT Ks
490    FOR Ns=L1 TO R1
500    F=Ns*(Dg(Ns+1)-Dg(Ns))
510    T=Ns*D2
520    FOR Ks=K1 TO K2
530    CALL Bessel(T*Ks,J0,J1,A,B0,B1)
540    Gw(Ks)=Gw(Ks)+F*B1
550    NEXT Ks
560    NEXT Ns
570    FOR Ks=K1 TO K2
580    Gw(Ks)=Gw(Ks)/(Ks*Delw)
590    NEXT Ks
600    PRINT Gw(*)
610    END
```

APPENDIX C

DERIVATION OF INTEGRATION RULE FOR PARABOLIC FITS TO g(x)

The situation of interest here is depicted in figure 2, where parabolas are fit to the samples of g(x), a pair of adjacent panels at a time. The derivation of the resultant approximation to integral (27) is broken down into the four cases illustrated in that figure.

It is again presumed, as in (20), that sample points of g(x) are taken at increment h, namely

$$x_n = nh \quad \text{for } \ell \le n \le r \ , \tag{C-1}$$

and that, in addition,

$$r - \ell \text{ is even } . \tag{C-2}$$

That is, the total number of panels employed in interval x_ℓ , x_r must be even. A breakdown of all the sample points $\{x_n\}$ into the four categories of figure 2 is depicted in figure C-1, where we have used the abbreviations

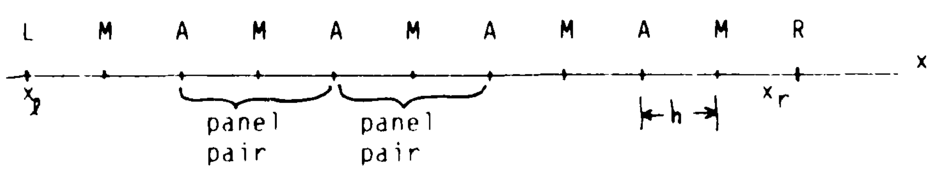

Figure C-1. Categorization of Sample Points

M = mid-point (of a panel pair),

A = abutting point (between two panel pairs),

L = left-end point,

R = right-end point. (C-3)

It is presumed in the following that $\omega > 0$; the case for $\omega = 0$ is given by (29), while $\omega < 0$ is immediately covered by observing that J_o is even.

Mid-Point

The contribution of a mid-point x_n to integral (27) is (see figure 2)

$$M_n = \int_{x_n-h}^{x_n+h} dx\ J_o(\omega x)\ g_n\ (1 - y^2) =$$

$$= \frac{g_n}{\omega} \int_{(n-1)\Theta}^{(n+1)\Theta} dt\ J_o(t) \left[1 - \left(\frac{t}{\Theta} - n\right)^2\right], \qquad \text{(C-4)}$$

where we utilized $t = \omega x$, (C-1), (28), and (22). Upon expansion of the square in (C-4), and use of (13)-(17), (C-4) reduces to the rather compact form

$$M_n = \frac{g_n}{\omega}\left\{(n^2 - 1)\ (B_1[(n-1)\theta] - B_1[(n+1)\theta]) - \right.$$

$$\left. - \frac{1}{\theta^2}\ (B_0[(n-1)\theta] - B_0[(n+1)\theta])\right\} . \qquad \text{(C-5)}$$

This type of term is yielded for $n = \ell + 1,\ \ell + 3,\ \dots,\ r - 3,\ r - 1$, as reference to figure C-1 will verify. Here, and in the following, for the sake of brevity, we do not document the rather detailed machinations that lead to the compact form (C-5) from the integral definition (C-4). The reader will have to reconstruct those nonprofitable manipulations, if interested.

At this juncture, instead of treating an abutting point with its associated 4 panels (see upper right of figure 2), we split it up into a panel pair with a left-end point and another panel pair with a right-end point. We thus have to consider a general left point and a general right point.

LEFT POINT

This case is obtained by looking at the bottom-left diagram in figure 2 and replacing ℓ by n everywhere. The contribution of this type of panel pair is

$$L_n = \int_{x_n}^{x_n+2h} dx\ J_0(\omega x)\ g_n\ (1 - y)\ (1 - y/2) =$$

$$= \frac{g_n}{\omega} \int_{n\Theta}^{(n+2)\Theta} dt\ J_0(t) \left[1 - \frac{3}{2}\left(\frac{t}{\Theta} - n\right) + \frac{1}{2}\left(\frac{t}{\Theta} - n\right)^2\right] =$$

$$= \frac{g_n}{2\omega}\Big\{(n+1)(n+2)\Big(B_1[(n+2)\Theta] - B_1[n\Theta]\Big) -$$

$$- \frac{1}{\Theta^2}\Big(B_0[(n+2)\Theta] - B_0[n\Theta]\Big) - 2\ J_1[n\Theta]\Big\} . \qquad \text{(C-6)}$$

This type of term is yielded for $n = \ell, \ell + 2, \ldots, r - 4, r - 2$, but not $n = r$; see figure C-1.

RIGHT POINT

This case pertains for the bottom-right diagram in figure 2 when r is replaced by n everywhere. The corresponding contribution to integral (27) is

$$R_n = \int_{x_n-2h}^{x_n} dx\ J_0(\omega x)\ g_n\ (1 + y)\ (1 + y/2) =$$

$$= \frac{g_n}{\omega} \int_{(n-2)\Theta}^{n\Theta} dt\ J_0(t) \left[1 + \frac{3}{2}\left(\frac{t}{\Theta} - n\right) + \frac{1}{2}\left(\frac{t}{\Theta} - n\right)^2\right] =$$

$$= \frac{g_n}{2\omega}\Big\{(n-1)(n-2)\Big(B_1[n\Theta] - B_1[(n-2)\Theta]\Big) -$$

$$- \frac{1}{\Theta^2}\Big(B_0[n\Theta] - B_0[(n-2)\Theta]\Big) + 2\ J_1[n\Theta]\Big\} . \qquad \text{(C-7)}$$

This type of term is yielded for $n = \ell + 2, \ell + 4, \ldots, r - 2, r$, but not $n = \ell$; see figure C-1.

ABUTTING POINT

We can now immediately obtain the integral contribution for an abutting point (top-right diagram of figure 2) by adding (C-6) and (C-7):

$$A_n = L_n + R_n =$$

$$= \frac{g_n}{2\omega}\Big\{(n+1)(n+2)\ B_1[(n+2)\Theta] - \frac{1}{\Theta^2}\ B_0[(n+2)\Theta] - 6n\ B_1[n\Theta] -$$

$$- (n-1)(n-2)\ B_1[(n-2)\Theta] + \frac{1}{\Theta^2}\ B_0[(n-2)\Theta]\Big\}\ , \qquad \text{(C-8)}$$

which holds only for $n = \ell + 2, \ell + 4, \ldots, r - 4, r - 2$; see figure C-1. As a notational shortcut, we say $n = (\ell + 2)(2)(r - 2)$ are the allowed values of n.

At this stage, we have succeeded in evaluating all the types of terms that have been depicted in figures 2 and C-1. The total approximation to integral (27) is therefore

$$G(\omega) \cong \sum_{n=\ell+1}^{r-1} M_n + \sum_{n=\ell+2}^{r-2} A_n + L_\ell + R_r\ , \qquad \text{(C-9)}$$

in terms of the contributions in (C-5)-(C-8), where the slash on the summation symbol denotes skipping every other term.

However, this grouping of terms in (C-9) is according to sample values $g_n = g(nh)$ of function $g(x)$. It is advantageous to re-arrange this sum, grouping terms instead according to sample values of functions $B_0(u)$ and $B_1(u)$, defined in (16) and (17). After considerable manipulations, the following alternative to (C-9) is obtained:

$$
\begin{aligned}
2\omega\, G(\omega) \cong\ & \frac{1}{\Theta^2} S_\ell\, B_0(\ell\Theta) - Q_\ell\, B_1(\ell\Theta) - 2g_\ell\, J_1(\ell\Theta) - \\
& - \frac{1}{\Theta^2} S_r\, B_0(r\Theta) + Q_r\, B_1(r\Theta) + 2g_r\, J_1(r\Theta) + \\
& + \frac{1}{\Theta^2} \sum_{n=\ell+2}^{r-2} D_n\, B_0(n\Theta) - \sum_{n=\ell+2}^{r-2} R_n\, B_1(n\Theta) \ .
\end{aligned}
\qquad \text{(C-10)}
$$

The auxiliary sequences utilized in (C-10) are defined below:

$$
\begin{aligned}
S_\ell &= g_{\ell+2} - 2g_{\ell+1} + g_\ell \\
S_r &= g_r - 2g_{r-1} + g_{r-2} \\
Q_\ell &= \ell(\ell+1)g_{\ell+2} - 2\ell(\ell+2)g_{\ell+1} + (\ell+2)(\ell+1)g_\ell \\
Q_r &= (r-2)(r-1)g_r - 2r(r-2)g_{r-1} + r(r-1)g_{r-2}
\end{aligned}
\qquad \text{(C-11)}
$$

and

$$
\left.
\begin{aligned}
D_n &= g_{n+2} - 2g_{n+1} + 2g_{n-1} - g_{n-2} \\
F_n &= g_{n+2} - 4g_{n+1} + 6g_n - 4g_{n-1} + g_{n-2} \\
R_n &= n^2 D_n + nF_n
\end{aligned}
\right\} \text{ for } n = (\ell+2)(2)(r-2) \ .
\qquad \text{(C-12)}
$$

It is important to observe from (C-10) that the Bessel integrals $B_0(u)$ and $B_1(u)$ need be evaluated only at $u = n\Theta$ for $n = \ell(2)r$, and need not be evaluated at the in-between points $n = (\ell + 1)(2)(r - 1)$. Of course, the input function $g(x)$ must be evaluated at all $x = x_n = nh$ for $n = \ell(1)r$. The quantities in (C-11) and (C-12) do not depend on $\Theta = \omega h$, and can be computed just once and stored, in preparation for use in (C-10).

If we are interested in evaluating integral $G(\omega)$ in (27) at values of ω equal to integer multiples k of some increment Δ, then we must substitute

$$\omega = k\Delta \quad \text{and} \quad \Theta = \omega h = k\Delta h \tag{C-13}$$

into (C-10). Then interest centers on computation of $B_0(u)$ and $B_1(u)$ at $u = m\Delta h$ for certain integers m. This consideration has been discussed in the sequel to (26).

BEHAVIOR FOR SMALL Θ

When Θ is small, differences of functions with similar values are required in (C-10). This same behavior obtains for Filon's method; see [5; (25.4.53)]. Accordingly, it is useful to have a series expansion for $G(\omega)$ about $\Theta = 0$, to be used for small Θ.

Since [5; (9.1.12)]

$$J_0(u) \sim 1 - \frac{1}{4} u^2 \quad \text{as } u \to 0 , \tag{C-14}$$

substitution in (C-4), along with the change of variable $y = t/\Theta - n$, yields the mid-point contribution

$$M_n = g_n\ h \int_{-1}^{1} dy\ J_0(\Theta(n+y))\ (1 - y^2) =$$

$$\sim g_n\ h \int_{-1}^{1} dy \left[1 - \frac{1}{4}\Theta^2(n+y)^2\right] (1 - y^2) =$$

$$= \frac{4}{3} g_n\ h\left\{1 - \frac{1}{4}\Theta^2\left(n^2 + \frac{1}{5}\right)\right\} \text{ as } \Theta \to 0\ . \tag{C-15}$$

A similar procedure for left point and right point contributions (C-6) and (C-7) gives

$$L_n = R_n \sim \frac{1}{3} g_n\ h\left\{1 - \frac{1}{4}\Theta^2(n^2 - \frac{2}{5})\right\} \text{ as } \Theta \to 0\ . \tag{C-16}$$

The total asymptotic contribution to $G(\omega)$ in (27) is therefore given by (a modified version of (C-9))

$$G(\omega) \sim \sum_{n=\ell}^{r-2} L_n + \sum_{n=\ell+2}^{r} R_n + \sum_{n=\ell+1}^{r-1} M_n \quad \text{as } \Theta \to 0\ , \tag{C-17}$$

using (C-15) and (C-16). For $\Theta = 0$, this reduces to Simpson's rule, (29). Additional correction terms involving Θ^4, Θ^6 could be derived by using additional terms in expansion (C-14).

When ω is specialized to values $\omega = k\Delta$ in (C-10), the result is as given in (36). Programs for both a faster high-storage procedure and a slower low-storage procedure are listed below.

```
10  ! ZERO-TH ORDER BESSEL TRANSFORM USING PARABOLIC INTERPOLATION.
20  ! INTEGRAL(Xl,Xr) dX Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
30  ! Gw(Ks), where W = Ks*Delw.       Faster high-storage.
40    Delx=.03                     !  INCREMENT (h) IN X
50    L=0                          !  Xl=L*Delx, L>=0
60    R=300                        !  Xr=R*Delx, R-L MUST BE EVEN & >=4
70    Delw=1.                      !  INCREMENT (Δ) IN W
80    K1=0                         !  W1=K1*Delw, K1>=0
90    K2=40                        !  W2=K2*Delw, K2>=K1
100   DOUBLE L,R,K1,K2,K0,L1,L2,R1,R2,Ns,Ks,I  !  INTEGERS
110   DIM Gx(890),Gw(500),Sq(500),J1l(500),J1r(500)
120   DIM B0(20000),B1(20000)
130   K0=K1
140   K1=MAX(K1,1)
150   L1=L+1
160   L2=L+2
170   R1=R-1
180   R2=R-2
190   REDIM Gx(L:R),Gw(K0:K2),Sq(K1:K2),J1l(K1:K2),J1r(K1:K2)
200   REDIM B0(L*K1:R*K2),B1(L*K1:R*K2)
210   FOR Ks=K0 TO K2
220   Gw(Ks)=0.
230   NEXT Ks
240   FOR Ns=L TO R
250   Gx(Ns)=FNG(Ns*Delx)          !  SEE DEF FNG(X) = g(X)
260   NEXT Ns
270   Gl=Gx(L)
280   Gr=Gx(R)
290   IF K0>0 THEN 380
300   Sl=S2=0.
310   FOR Ns=L1 TO R1 STEP 2
320   Sl=Sl+Gx(Ns)
330   NEXT Ns
340   FOR Ns=L2 TO R2 STEP 2
350   S2=S2+Gx(Ns)
360   NEXT Ns
370   Gw(0)=(Gl+Gr+4.*Sl+2.*S2)*Delx/3.
380   Gl1=Gx(L1)
390   Gl2=Gx(L2)
400   Gr1=Gx(R1)
410   Gr2=Gx(R2)
420   Sl=Gl2-2.*Gl1+Gl
430   Sr=Gr-2.*Gr1+Gr2
440   Ol=L*L1*Gl2-2.*L*L2*Gl1+L2*L1*Gl
450   Or=R2*R1*Gr-2.*R*R2*Gr1+R*R1*Gr2
460   Gl2=Gl*2.
470   Gr2=Gr*2.
480   D2=Delw*Delx
490   FOR Ks=K1 TO K2
500   F=Ks*D2
510   Sq(Ks)=1./(F*F)
520   NEXT Ks
```

```
530   IF L=0 THEN 580
540   FOR Ks=K1 TO K2
550   I=L*Ks
560   CALL Bessel(I*D2,J0,J1l(Ks),A,B0(I),B1(I))
570   NEXT Ks
580   FOR Ks=K1 TO K2
590   I=R*Ks
600   CALL Bessel(I*D2,J0,J1r(Ks),A,B0(I),B1(I))
610   NEXT Ks
620   FOR Ns=L2 TO R2 STEP 2
630   FOR Ks=K1 TO K2
640   I=Ns*Ks
650   IF B0(I)<>0. THEN 670
660   CALL Bessel(I*D2,J0,J1,A,B0(I),B1(I))
670   NEXT Ks
680   NEXT Ns
690   IF L=0 THEN 740
700   FOR Ks=K1 TO K2
710   I=L*Ks
720   Gw(Ks)=Sq(Ks)*Sl*B0(I)-Ql*B1(I)-Gl2*J1l(Ks)
730   NEXT Ks
740   FOR Ks=K1 TO K2
750   I=R*Ks
760   F=Sq(Ks)*Sr*B0(I)-Qr*B1(I)-Gr2*J1r(Ks)
770   Gw(Ks)=Gw(Ks)-F
780   NEXT Ks
790   FOR Ns=L2 TO R2 STEP 2
800   G2=Gx(Ns+2)
810   G1=Gx(Ns+1)
820   H1=Gx(Ns-1)
830   H2=Gx(Ns-2)
840   Dn=G2-2.*G1+2.*H1-H2
850   Fn=G2-4.*G1+6.*Gx(Ns)-4.*H1+H2
860   Rn=Ns*(Ns*Dn+Fn)
870   FOR Ks=K1 TO K2
880   I=Ns*Ks
890   Gw(Ks)=Gw(Ks)+Sq(Ks)*Dn*B0(I)-Rn*B1(I)
900   NEXT Ks
910   NEXT Ns
920   F=Delw*2.
930   FOR Ks=K1 TO K2
940   Gw(Ks)=Gw(Ks)/(Ks*F)
950   NEXT Ks
960   PRINT Gw(*)
970   PAUSE
980   END
990   !
1000  DEF FNG(X)                      !  g(X)
1010    Gx=X*EXP(-.5*X*X)             !  RAYLEIGH EXAMPLE
1020  RETURN Gx
1030  FNEND
```

```
 10  ! ZERO-TH ORDER BESSEL TRANSFORM USING PARABOLIC INTERPOLATION.
 20  ! INTEGRAL(Xl,Xr) dX Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
 30  ! Gw(Ks), where W = Ks*Delw.      Slower low-storage.
 40    Delx=.03                      !  INCREMENT (h) IN X
 50    L=0                           !  Xl=L*Delx, L>=0
 60    R=300                         !  Xr=R*Delx, R-L MUST BE EVEN & >=4
 70    Delw=1.                       !  INCREMENT (Δ) IN W
 80    K1=0                          !  W1=K1*Delw, K1>=0
 90    K2=120                        !  W2=K2*Delw, K2>=K1
100    DOUBLE L,R,K1,K2,K0,L1,L2,R1,R2,Ns,Ks   !  INTEGERS
110    DIM Gx(800),Gw(500),Sq(500)
120    K0=K1
130    K1=MAX(K1,1)
140    L1=L+1
150    L2=L+2
160    R1=R-1
170    R2=R-2
180    REDIM Gx(L:R),Gw(K0:K2),Sq(K1:K2)
190    FOR Ks=K0 TO K2
200    Gw(Ks)=0.
210    NEXT Ks
220    FOR Ns=L TO R
230    Gx(Ns)=FNG(Ns*Delx)            !  SEE DEF FNG(X) = g(X)
240    NEXT Ns
250    Gl=Gx(L)
260    Gr=Gx(R)
270    IF K0>0 THEN 360
280    S1=S2=0.
290    FOR Ns=L1 TO R1 STEP 2
300    S1=S1+Gx(Ns)
310    NEXT Ns
320    FOR Ns=L2 TO R2 STEP 2
330    S2=S2+Gx(Ns)
340    NEXT Ns
350    Gw(0)=(Gl+Gr+4.*S1+2.*S2)*Delx/3.
360    Gl1=Gx(L1)
370    Gl2=Gx(L2)
380    Gr1=Gx(R1)
390    Gr2=Gx(R2)
400    Sl=Gl2-2.*Gl1+Gl
410    Sr=Gr-2.*Gr1+Gr2
420    Dl=L*L1*Gl2-2.*L*L2*Gl1+L2*L1*Gl
430    Dr=R2*R1*Gr-2.*R*R2*Gr1+R*R1*Gr2
440    Gl2=Gl*2.
450    Gr2=Gr*2.
460    D2=Delw*Delx
470    FOR Ks=K1 TO K2
480    F=Ks*D2
490    Sq(Ks)=1./(F*F)
500    NEXT Ks
```

```
510     IF L=0 THEN 570
520     T=L*D2
530     FOR Ks=K1 TO K2
540     CALL Bessel(T*Ks,J0,J1,A,B0,B1)
550     Gw(Ks)=Sq(Ks)*S1*B0-O1*B1-G12*J1
560     NEXT Ks
570     T=R*D2
580     FOR Ks=K1 TO K2
590     CALL Bessel(T*Ks,J0,J1,A,B0,B1)
600     F=Sq(Ks)*Sr*B0-Or*B1-Gr2*J1
610     Gw(Ks)=Gw(Ks)-F
620     NEXT Ks
630     FOR Ns=L2 TO R2 STEP 2
640     G2=Gx(Ns+2)
650     G1=Gx(Ns+1)
660     H1=Gx(Ns-1)
670     H2=Gx(Ns-2)
680     Dn=G2-2.*G1+2.*H1-H2
690     Fn=G2-4.*G1+6.*Gx(Ns)-4.*H1+H2
700     Rn=Ns*(Ns*Dn+Fn)
710     T=Ns*D2
720     FOR Ks=K1 TO K2
730     CALL Bessel(T*Ks,J0,J1,A,B0,B1)
740     Gw(Ks)=Gw(Ks)+Sq(Ks)*Dn*B0-Rn*B1
750     NEXT Ks
760     NEXT Ns
770     F=Delw*2.
780     FOR Ks=K1 TO K2
790     Gw(Ks)=Gw(Ks)/(Ks*F)
800     NEXT Ks
810     PRINT Gw(*)
820     PAUSE
830     END
840     !
850     DEF FNG(X)                          !   g(X)
860       Gx=X*EXP(-.5*X*X)                 !   RAYLEIGH EXAMPLE
870     RETURN Gx
880     FNEND
```

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